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The orbital stability of periodic motions of a Hamiltonian system with two degrees of freedom in the case of 3:1 resonance☆

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Abstract

The problem of the orbital stability of periodic motions, produced from an equilibrium position of an autonomous Hamiltonian system with two degrees of freedom is considered. The Hamiltonian function is assumed to be analytic and alternating in a certain neighbourhood of the equilibrium position, the eigenvalues of the matrix of the linearized system are pure imaginary, and the frequencies of the linear oscillations satisfy a 3:1 ratio. The problem of the orbital stability of periodic motions is solved in a rigorous non-linear formulation. It is shown that short-period motions are orbitally stable with the sole exception of the case corresponding to bifurcation of short-period and long-period motions. In this particular case there is an unstable short-period orbit. It is established that, if the equilibrium position is stable, then, depending on the values of the system parameters, there is only one family of orbitally stable long-period motions, or two families of orbitally stable and one family of unstable long-period motions. If the equilibrium position is unstable, there is only one family of unstable long-period motions or one family of orbitally stable and two families of unstable long-period motions. Special cases, corresponding to bifurcation of long-period motions or degeneration in the problem of stability, when an additional analysis is necessary, may be exceptions. The problem of the orbital stability of the periodic motions of a dynamically symmetrical satellite close to its steady rotation is considered as an application.

1. Formulation of the problem

Consider a system with two degrees of freedom, the motion of which is described by the Hamilton equations

$$\frac{dq_j}{dt} = \frac{\partial H}{\partial p_j}, \quad \frac{dp_j}{dt} = -\frac{\partial H}{\partial q_j}; \quad j = 1, 2 \tag{1.1}$$

Suppose the origin of coordinates $q_j = p_j = 0$ ($j = 1, 2$) of phase space is an equilibrium position, while the Hamiltonian function H is analytic in a certain part of its neighbourhood and depends explicitly on the time t.

We will assume that the roots $\pm i\omega_j$ ($\omega_j > 0, j = 1, 2$) of the characteristic equation of the linearized system (1.1) are pure imaginary, while the quadratic part of the Hamiltonian function is alternating. We will assume, moreover, that there is exact resonance of the fourth order, i.e. the frequencies of linear oscillations are connected by the relation $\omega_1 = 3\omega_2$. In this case the canonical variables q_j, p_j ($j = 1, 2$) can be chosen so that the Hamiltonian function does not contain terms of the third and fifth powers, while terms of the second, fourth and sixth powers are reduced to normal form.

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With this choice of variables, the Hamiltonian H has the form[1]

$$H = \frac{1}{2}\omega_1(q_1^2+p_1^2) - \frac{1}{2}\omega_2(q_2^2+p_2^2) + H_4 + H_6 + O_7 \tag{1.2}$$

$$\begin{aligned} H_4 = {} & \frac{1}{4}c_{20}(q_1^2+p_1^2)^2 + \frac{1}{4}c_{11}(q_1^2+p_1^2)(q_2^2+p_2^2) + \frac{1}{4}c_{02}(q_2^2+p_2^2)^2 + \\ & + \frac{1}{36}\sqrt{3}b(q_2^2(q_1q_2-3p_2p_1) + p_2^2(p_2p_1-3q_1q_2)) \end{aligned} \tag{1.3}$$

$$H_6 = \frac{1}{4}\sum_{s+l=3} c_{sl}(q_1^2+p_1^2)^s(q_2^2+p_2^2)^l + \tilde{H}_6 \tag{1.4}$$

where b, c_{sl} $(s+l=2,\ 3)$ are constant coefficients. We will further assume that the limitations $b \neq 0$ and $c_{20}+3c_{11}+9c_{02} \neq 0$ are satisfied. The form $\tilde{H}_6$ contains resonance terms of the sixth power in q_j and p_j, the explicit form of which will not be further necessary. Here and henceforth we will denote by O_n the series in powers of canonical variables beginning with terms of power no less than n.

We will make the canonical replacement of variables $q_j,\ p_j \to x_j,\ y_j$ in accordance with the formulae $q_i = \sqrt{\varepsilon\delta\omega_2/c}x_i,\ p_i = \sqrt{\varepsilon\delta\omega_2/c}y_i,\ c = c_{20}+3c_{11}+9c_{02},\ \delta = \text{sign}(\omega_2/c)$, where $\varepsilon(\varepsilon \ll 1)$ is a quantity which defines the order of smallness of the neighbourhood of the equilibrium position considered.

If, moreover, we introduce the new independent variable $\tau = \omega_2 t$, Hamiltonian (1.2) takes the form

$$\begin{aligned} H = {} & \frac{3}{2}r_1^2 - \frac{1}{2}r_2^2 + \varepsilon\frac{1}{4}\Big\{a_{20}r_1^4 + a_{11}r_1^2r_2^2 + a_{02}r_2^4 + \\ & + \frac{\sqrt{3}}{9}\gamma[x_2^2(x_1x_2-3y_2y_1) + y_2^2(y_2y_1-3x_1x_2)]\Big\} + O(\varepsilon^2) \end{aligned} \tag{1.5}$$

where

$$r_j^2 = x_j^2 + y_j^2, \quad j = 1, 2, \quad \gamma = \frac{\delta l}{c}, \quad a_{ij} = \frac{\delta c_{ij}}{c}, \quad i+j = 2 \tag{1.6}$$

Note that the coefficients a_{ij} are connected by the relation $a_{20}+3a_{11}+9a_{02} = \delta$.

In this paper, we solve, in a rigorous non-linear formulation, the problem of the orbital stability of the periodic motions of system (1.1), produced from the equilibrium position. A non-linear analysis of the orbital stability of periodic motions was carried out previously[2–5] for resonances of the first, second and third orders. We use a similar investigation procedure below.

2. The stability of short-period motions

The system of equations (1.1) possesses families of periodic motions, created from the equilibrium position. In this section we investigate, in a rigorous non-linear formulation, the orbital stability of the periodic motions, the period of which with respect to τ is close to $2\pi/3$. In the terminology employed, motions of this type will be called short-period motions, unlike the long-period motions whose period with respect to τ is close to 2π (see Section 3). The existence of a family of short-period motions follows from Lyapunov's theorem on the holomorphic integral.[6] On the basis of this theorem it can also be asserted[7,8] that, by an almost identical canonical transformation of the variables $x_j, y_j \to x'_j,\ y'_j\ (j = 1, 2)$, which converges for fairly small ε, Hamiltonian (1.5) can be reduced to the form

$$H = H_0(r_1'^2, \varepsilon) + O(r_2'^2), \quad r_j'^2 = x_j'^2 + y_j'^2$$

where H_0 depends only on r'^2_2 and on the small parameter ε. This transformation does not change the form of terms of order ε^2 inclusive, written explicitly in Hamiltonian (1.5).

To investigate the orbital stability of short-period motions we will change to the variablesφ_1, R_1, X_2, Y_2 using the formulae

$$x_1' = \sqrt{2R_1}\sin\varphi_1, \quad y_1' = \sqrt{2R_1}\cos\varphi_1, \quad x_2' = X_2, \quad y_2' = Y_2$$

In these variables the short-period motions are described by the family of solutions

$$\varphi_1 = \Omega_1(\tau+\tau_0), \quad R_1 = \frac{C}{2}, \quad X_2 = Y_2 = 0 \tag{2.1}$$

where $C>0$ and τ_0 are arbitrary constants, the first of which represents the amplitude of the short-period motion. The frequency Ω_1 depends on the amplitude C and can be represented in the form

$$\Omega_1 = 3 + \varepsilon a_{20}C + \frac{3}{4}\varepsilon^2 a_{30}C^2 + O(\varepsilon^2) \tag{2.2}$$

We will introduce the perturbations ρ_1, ξ_2, η_2

$$R_1 = \frac{C}{2} + \rho_1, \quad X_2 = \xi_2, \quad Y_2 = \eta_2$$

The Hamiltonian of the perturbed motion can be represented in the form of a converging series

$$\Gamma = \Gamma_2 + \Gamma_3 + \ldots + \Gamma_m \ldots \tag{2.3}$$

where Γ_m is the form of power m in $\sqrt{|\rho_1|}$, ξ_2, η_2 with coefficients that are 2π-periodic in φ_1. The forms Γ_m depend analytically on the small parameter ε. For $m=2, 3, 4$ their expansions in series in powers of ε havethe form

$$\Gamma_2 = \Omega_1\rho_1 + \frac{1}{2}(\xi_2^2 + \eta_2^2) + \varepsilon\Gamma_2^*(\xi_2, \eta_2, \phi_1, \varepsilon)$$

$$\Gamma_3 = \varepsilon\frac{\sqrt{3}\gamma C}{108}[(\xi_2^3 - 3\xi_2\eta_2^2)\sin\varphi_1 + (\eta_2^3 - 3\eta_2\xi_2^2)\cos\varphi_1] + O(\varepsilon^2) \tag{2.4}$$

$$\Gamma_4 = \varepsilon\frac{1}{3}\left[a_{20}\rho_1^2 + \frac{1}{2}a_{11}(\xi_2^2+\eta_2^2)\rho_1 + \frac{1}{4}a_{02}(\xi_2^2+\eta_2^2)^2\right] + O(\varepsilon^2)$$

where $\Gamma_2^*(\xi_2, \eta_2, \phi_1, \varepsilon)$ is the quadratic form of the variables ξ_2, η_2, the coefficients of which are 2π-periodic functions of the variable ϕ_1 and depend analytically on the small parameter ε.

The problem of the orbital stability of the short-period motions is equivalent to the problem of the stability of the canonical system with Hamiltonian (2.3) with respect to the variables ρ_1, ξ_2, η_2.

Following the procedure employed to analyse the stability of Hamiltonian systems,[1] we obtain the normal form of the Hamiltonian (2.3) and then apply the results of the KAM theory.[8,9]

Since, for any integer n, the inequality $n\Omega_1 \neq 2$ is satisfied, which guarantees that there is no parametric resonance, then, using the canonical univalent replacement of variables φ_1, ρ_1, $\xi_2\,\eta_2 \to \tilde{\varphi}_1$, $\tilde{\rho}_1$, $\tilde{\xi}_2\,\tilde{\eta}_2$, which is real, analytic in ε close to identical, 2π-periodic in φ_1 and linear in ξ_2, η_2 (see, for example, Ref. 1) the Hamiltonian function (2.3) can be reduced to the form

$$\tilde{\Gamma} = \Omega_1\tilde{\rho}_1 - \frac{1}{2}\Omega_2(\tilde{\xi}_2^2 + \tilde{\eta}_2^2) + \tilde{\Gamma}_3 + \tilde{\Gamma}_4 + \ldots + \tilde{\Gamma}_m + \ldots \tag{2.5}$$

The forms $\tilde{\Gamma}_3$ and $\tilde{\Gamma}_4$, apart from terms of the order of ε^2, are identical with Γ_3 and Γ_4, while the part of the new Hamiltonian (2.5), quadratic in the variables $|\tilde{\rho}_1|^{1/2}$, $\tilde{\xi}_2$, $\tilde{\eta}_2$ is written in normal form, where the quantity Ω_2 depends analytically on the small parameter ε. The expansion of Ω_2 in series in powers of ε, calculated taking into account terms up to the sixth order inclusive in the initial Hamiltonian (1.2), has the form

$$\Omega_2 = 1 - \frac{1}{2}\varepsilon a_{11}C - \frac{1}{4}\varepsilon^2 a_{21}C^2 + O(\varepsilon^3) \tag{2.6}$$

We can normalize terms of the third and fourth power of Hamiltonian (2.5), for example, by the Deprit–Hori method[1,10] The canonical normalizing transformation $\tilde{\varphi}_1, \tilde{\rho}_1, \tilde{\xi}_2, \tilde{\eta}_2 \to \omega_1, I_1, u_2, v_2$ will be real, close to identical, analytic in I_1, u_2, v_2 and 2π-periodic in w_1.

Since $\Omega_1 \approx 3$ and $\Omega_2 \approx 1$, when the equality $\Omega_1 \neq 3\Omega_2$ is satisfied in the system with Hamiltonian (2.5) resonances of lower orders are impossible and the normalized Hamiltonian has the form

$$\Gamma_* = \Omega_1 I_1 - \Omega_2 I_2 + C_{20} I_1^2 + C_{11} I_1 I_2 + C_{02} I_2^2 + O_3, \quad I_2 = (u_2^2 + v_2^2)/2 \tag{2.7}$$

Without dwelling on detailed calculations, we will write expressions for the coefficients of the normal form

$$C_{11} = -\frac{\gamma^2}{18\kappa^2} + O(\varepsilon), \quad C_{20} = \varepsilon a_{20} + O(\varepsilon^2), \quad C_{02} = \varepsilon a_{02} + O(\varepsilon^2) \tag{2.8}$$

where $\kappa = 3a_{11} + 2a_{20}$; the case $\kappa = 0$ is not considered in this paper.

On the basis of the Arnol'd-Moser theorem,[8,9] when the condition for isoenergetic non-degeneracy

$$\Delta_1 = C_{20}\Omega_2^2 + C_{11}\Omega_1\Omega_2 + C_{02}\Omega_1^2 = -\frac{\gamma^2}{6\kappa^2} + O(\varepsilon) \neq 0$$

is satisfied, the system with Hamiltonian (2.3) is Lyapunov stable. Since, in view of the limitation imposed above, $\gamma \neq 0$, for fairly small ε the inequality $\Delta_1 \neq 0$ holds and the short-period motion is orbitally stable.

Suppose now that the resonance relation $\Omega_1 = 3\Omega_2$ is satisfied. We will show that in this case instability of the short-period motion occurs. For this purpose we consider the motion of a system with Hamiltonian (2.5) at the energy level $\tilde{\Gamma} = 0$.

The coordinate $\tilde{\varphi}_1$ is an increasing function of time, and hence to describe the motion at a fixed energy level we can take the independent variable as the new variable. From the equation $\tilde{\Gamma} = 0$ for small $\tilde{\xi}_2, \tilde{\eta}_2, \tilde{\rho}_1$ we have $\tilde{\rho}_1 = -K(\tilde{\xi}_2, \tilde{\eta}_2, \tilde{\varphi}_1)$. The canonical system (Whittaker's equation) with Hamiltonian K and independent variable $\tilde{\varphi}_1$ describes the evolution of the variables ξ_2, η_2 at the zero isoenergy level. The expansion of $K(\tilde{\xi}_2, \tilde{\eta}_2, \tilde{\varphi}_1)$ in series in powers of ξ_2, η_2 has the form

$$K = \frac{1}{2}\lambda(\tilde{\xi}_2^2 + \tilde{\eta}_2^2) + K_3 + \ldots + K_m + \ldots \tag{2.9}$$

where

$$\begin{aligned} \lambda &= -\frac{\Omega_2}{\Omega_1} = -\frac{1}{3}\left(1 - \varepsilon C\frac{\kappa}{6} + \varepsilon^2 C^2 \chi + O(\varepsilon^3)\right) \\ \chi &= \frac{1}{18}a_{20}\kappa - \frac{1}{4}(a_{30} + a_{21}), \quad a_{ij} = \frac{c_{ij}\omega_2}{c^2}, \quad i + j = 3 \end{aligned} \tag{2.10}$$

and K_m is a form of power m in $\tilde{\xi}_2, \tilde{\eta}_2$ with coefficients that are 2π-periodic in $\tilde{\varphi}_1$ and analytic with respect to ε.

It follows from relations (2.10) that the resonance case $\lambda = -1/3$ considered is only possible for fairly small κ, namely when $\kappa \sim \varepsilon$, and hence we will henceforth put $\kappa = \varepsilon\kappa_0$, where $\kappa_2 \neq 0$. Then by the implicit-function theorem there exists furthermore a unique value $C = C_*$ of the amplitude of the short-period motion, for which the resonance relation $\lambda = -1/3$ holds. We have the following expression for C_*

$$C_* = \frac{\kappa_0}{6\chi} + O(\varepsilon) \tag{2.11}$$

Note that this value of the amplitude corresponds to bifurcations of the short-period and long-period motions. Hence, when $C = C_*$ a third-order resonance occurs in the system with Hamiltonian (2.9). In this case a canonical, analytic in

ε, 2π-periodic in $\tilde{\varphi}_1$ replacement of variables exists which converts Hamiltonian (2.9) to the following normal form[1]

$$F = -\frac{1}{6}(u_2^2+v_2^2)+f_{30}(\tilde{\varphi}_1)(u_2^3-3u_2v_2^2)+f_{03}(\tilde{\varphi}_1)(v_2^3-3v_2u_2^2)+O_4$$
$$f_{30} = d_{30}\cos\tilde{\varphi}_1-d_{03}\sin\tilde{\varphi}_1,\quad f_{03} = d_{30}\sin\tilde{\varphi}_1+d_{30}\cos\tilde{\varphi}_1 \tag{2.12}$$

The coefficients d_{30} and d_{03} depend analytically on ε, where we have the following estimates

$$d_{03} = -\varepsilon\frac{\sqrt{3}\gamma\kappa_0}{324\chi}+O(\varepsilon^2),\quad d_{03} = O(\varepsilon^2)$$

According to the results of an investigation of the stability of non-autonomous Hamiltonian systems with a single degree of freedom for resonances[1], a system with Hamiltonian (2.12) is unstable when $d_{30}^2+d_{03}^2\neq 0$. Since, for fairly small ε this condition is obviously satisfied, short-period motion with amplitude $C=C_*$ is orbitally unstable.

3. Investigation of the stability of long-period motions in the linear approximation

As has already been noted, in addition to the family of short-period motions, system (1.1) with Hamiltonian (1.5) possesses families of long-period motions, which are also created from the equilibrium position, but have a period with respect to τ close to 2π. In the resonance case considered, the existence of long-period motions does not follow from Lyapunov's theorem on the holomorphic integral. The problem of the existence of long-period motions in the case of fourth-order resonance has been investigated in Refs 11–13. The most complete results were obtained by Schmidt,[13] who also considered the linear problem of the orbital stability of long-period motions in the case of a positive-definite quadratic part of the initial Hamiltonian. Orbital stability of long-period motions in the case of an alternating quadratic part of the Hamiltonian has not been previously considered.

Long-period motions can be constructed in the form of converging series in powers of the small parameter ε.[11–13] In this case the first approximation is the solution of the truncated system with Hamiltonian

$$H_* = 3R_1-R_2+\varepsilon\left\{a_{20}R_1^2+a_{11}R_1R_2+a_{02}R_2^2+\frac{\gamma}{3\sqrt{3}}R_2\sqrt{R_1R_2}\cos(3\varphi_2+\varphi_1)\right\} \tag{3.1}$$

which is obtained from Hamiltonian (1.5) if terms of the order of ε^2 and higher are dropped, and a canonical replacement of variables is then carried out using the formulae

$$x_i = \sqrt{2R_i}\sin\varphi_i,\quad y_i = \sqrt{2R_i}\cos\varphi_i$$

In addition to the energy integral $H_*=\text{const}$, the truncated system has one other first integral $3R_1\text{-}R_2=\text{const}$. Bearing this in mind, we introduce the new canonical variables

$$\psi = \varphi_1,\quad \theta = \varphi_1+3\varphi_2,\quad J = R_1-\frac{R_2}{3},\quad R = \frac{R_2}{3} \tag{3.2}$$

In these variables the truncated Hamiltonian takes the form

$$H_* = 3J+\varepsilon\{a_{20}J^2+\kappa JR+\delta R^2+\gamma R\sqrt{R(J+R)}\cos\theta\} \tag{3.3}$$

The quantities δ and κ were introduced in Sections 1 and 2 respectively. Without loss of generality we can assume $\delta=1$. The variable ψ is a cyclic coordinate, while the momentum J corresponding to it is the first integral of the truncated system. We will further put $J=J_0=\text{const}$. The change in the variables θ and R is described by the canonical system of equations

$$\frac{d\theta}{d\tau} = \varepsilon\left\{\kappa J_0+2R+\gamma\frac{(4R+3J_0)\sqrt{R}}{2\sqrt{J_0+R}}\cos\theta\right\},\quad \frac{dR}{d\tau} = \varepsilon\gamma R\sqrt{R(J_0+R)}\sin\theta \tag{3.4}$$

in which J_0 plays the role of the parameter.

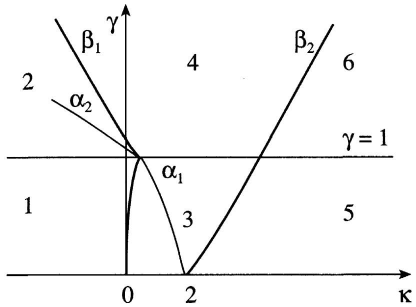

Fig. 1.

In system (3.4) there are equilibrium positions

$$\theta_0 = \frac{\pi}{2}[1 + \mathrm{sign}(J_0\gamma(\kappa + 2x))], \quad R_0 = xJ_0, \tag{3.5}$$

corresponding to periodic solutions of the truncated system. The quantity x in (3.5) is the real root of the equation

$$(16\gamma^2 - 16)x^3 + (24\gamma^2 - 16\kappa - 16)x^2 + (9\gamma^2 - 16\kappa - 4\kappa^2)x - 4\kappa^2 = 0 \tag{3.6}$$

Equation (3.6) is invariant under the replacement $\gamma \to -\gamma$, and hence, without loss of generality, we can put $\gamma > 0$.

We will investigate how the roots of Eq. (3.6) depend on the parameters κ and γ. We first note that when $\gamma = 1$ this equation degenerates into a quadratic equation. Suppose $\gamma \neq 0$. A simple analysis then shows that for values of the parameters which satisfy the equation

$$-16\kappa^4 + 96\kappa^3 + 72\kappa^2\gamma^2 - 192\kappa^2 - 72\gamma^2\kappa + 128\kappa - 36\gamma^2 + 27\gamma^4 = 0 \tag{3.7}$$

Eq. (3.6) has a multiple root. Equation (3.7) defines two curves β_1 and β_2 in the half-plane $\gamma > 0$ (see Fig. 1; the curves α_1 and α_2 are dealt with in Section 4). The curves β_1 and β_2 and the straight line $\gamma = 1$ split the half-plane $\gamma > 0$ into six regions, in which Eq. (3.6) has a fixed number of real roots. In regions 1, 2, 5 and 6 Eq. (3.6) has three different real roots, which we will denote by x_0, x_1 and x_2, while in regions 3 and 4 it has one real root x_0. The root x_0 depends continuously on the parameter κ, and a continuous dependence on γ occurs in the intervals (0;1) and (1; +∞). The roots x_1 and x_2 depend continuously on the parameter in the regions in which it exists and on their boundaries, with the sole exception of the boundary separating regions 1 and 2. On the boundaries β_1 and β_2 the roots x_1 and x_2 merge, forming a multiple root, which disappears on passing into regions 3 and 4. Henceforth, to fix our ideas, we will put $x_1 < x_2$.

On the boundary of regions 1 and 2 ($\gamma = 1$ and $\kappa \in (-\infty; 1/2)$) Eq. (3.6) has two different real roots $x^{(1)}$ and $x^{(2)}$ ($x^{(1)} < x^{(2)}$), where

$$x^{(1)} = \lim_{\gamma \to 1-0} x_0, \quad x^{(2)} = \lim_{\gamma \to 1+0} x_0 \tag{3.8}$$

On the boundary of regions 3 and 4 ($\gamma = 1$ and $\kappa \in (1/2; 9/2)$) there are no real roots, while on the boundary of regions 5 and 6 ($\gamma = 1$ and $\kappa \in (9/2; +\infty)$) Eq. (3.6) has two different real roots, which are the roots x_1 and x_2 also from regions 5 and 6, that transfer continuously through the boundary.

In Fig. 2 we show graphs of the roots x_0, x_1 and x_2 against the parameter κ (for fixed γ) for the cases $0 < \gamma < 1$ and $\gamma > 1$.

Note also that the boundary curves β_1 and β_2 may specify the following equations parametrically

$$\kappa = \frac{2s(2s+3)}{8s^2 + 12s + 3}, \quad \gamma^2 = \frac{64(s^3 + 3s^2 + 3s + 1)s}{(8s^2 + 12s + 3)^2} \tag{3.9}$$

The parameter s is the multiple root of Eq. (3.6). It takes the values in the intervals $(-\infty; -\sqrt{3}(1+\sqrt{3})/4) \cup (0; +\infty)$ and $(-\sqrt{3}(1+\sqrt{3})/4; -1)$ on curves β_1 and β_2 respectively.

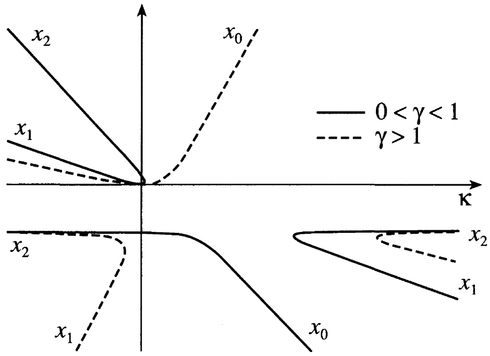

Fig. 2.

A family of long-period motions of the truncated system with Hamiltonian (3.1), which depends on the parameter J_0, corresponds to each root of Eq. (3.6). The long-period motions of the whole system can be constructed using the approach described in Refs 2 and 3. To do this it is necessary to carry out an isoenergetic reduction and transfer to Whittaker equations, using the coordinate ψ as the independent variable, and then use the Poincaré small-parameter method to construct periodic solutions of the reduced system, created from the equilibrium position. The long-period motions of the complete system, constructed using this method, in the variables ψ, θ, J, R will have the form

$$J_* = J_0 + O(\varepsilon), \quad \theta_* = \theta_0 + O(\varepsilon), \quad R_* = R_0 + O(\varepsilon) \tag{3.10}$$

The terms $O(\varepsilon)$ are series in powers of ε with coefficients that are 2π-periodic in ψ.

We will make a replacement of variables in accordance with the formulae

$$u = \sqrt{2R}\sin\theta, \quad \upsilon = \sqrt{2R}\cos\theta$$

where the pair of variables ψ, J undergo an identical transformation but the old notation is retained for them. The long-period motions in the variables ψ, J, ξ, η have the form

$$J_* = J_0 + O(\varepsilon), \quad u_* = O(\varepsilon), \quad \upsilon_* = \upsilon_0 + O(\varepsilon); \quad \upsilon_0 = \sqrt{2R_0}\cos\theta_0 \tag{3.11}$$

We will introduce the perturbations ρ, ξ, η

$$J = J_* + \rho, \quad u = u_* + \xi, \quad \upsilon = \upsilon_* + \eta \tag{3.12}$$

and expand the Hamiltonian of the perturbed motion in series in powers of ρ, ξ, η

$$\Gamma = \Gamma_2 + \Gamma_3 + \Gamma_4 + O_5 \tag{3.13}$$

where Γ_m is a form of power m in $\sqrt{|\rho|}$, ξ, η with coefficients that are 2π-periodic in φ_1 and analytic in ε. The expressions for Γ_m ($m = 2, 3, 4$) have the form

$$\begin{aligned} \Gamma_2 &= \Omega_1\rho + \varepsilon h_{20}\xi^2 + \varepsilon h_{02}\eta^2 + O(\varepsilon^2) \\ \Gamma_3 &= b_1\eta\rho + h_{21}\eta\xi^2 + h_{30}\eta^3 + O(\varepsilon^2) \\ \Gamma_4 &= b_2\rho^2 + b_3\rho\xi^2 + b_4\rho\eta^2 + h_{40}\xi^4 + h_{22}\xi^2\eta^2 + h_{04}\eta^4 + O(\varepsilon^2) \end{aligned} \tag{3.14}$$

where

$$\Omega_1 = 3 - \varepsilon(\gamma\upsilon_0^3(\upsilon_0^2 + J_0)/d_1 + \upsilon_0^4 - 4J_0^4a_{20})/(2J_0)$$

$$h_{20} = -\gamma\upsilon_0 d_1/4,\quad h_{02} = \upsilon_0(\gamma(2\upsilon_0^4 + 6\upsilon_0^2 J_0 + 3J_0^2)/d_1^3 + 2\upsilon_0)$$

$$h_{21} = \sqrt{3}\gamma(3\upsilon_0^2 - J_0)/(4d_2) + \upsilon_0,\quad h_{30} = \sqrt{3}\gamma(6\upsilon_0^4 - 6\upsilon_0^2 J_0 + J_0^2)/(2d_2^3) + \upsilon_0$$

$$h_{40} = 1/4 + 3\sqrt{3}\gamma\upsilon_0/(32d_2),\quad h_{22} = 1/2 - 9\sqrt{3}\gamma\upsilon_0(J_0 - \upsilon_0^2)/(8d_2^3) \tag{3.15}$$

$$h_{04} = 1/4 + 3\sqrt{3}\gamma\upsilon_0(6\upsilon_0^2 - 10\upsilon_0^2 J_0 + 5J_0^2)/(8d_2^3)$$

$$d_1 = \sqrt{\upsilon_0^2 + 2J_0},\quad d_2 = \sqrt{3\upsilon_0^2 - 2J_0}$$

Explicit expressions for the constant coefficients $b_1, \ldots, b_4$ will not be necessary in what follows.

In a linear system with Hamiltonian Γ_2 the equations corresponding to two pairs of canonical variables can be considered separately. The first pair of canonical variables describes harmonic oscillations with frequency Ω_1, and hence to solve the problem of the stability of a linear system it is sufficient to analyse the characteristic equation of the canonical system with Hamiltonian $\Gamma_2^* = h_{20}\xi^2 + h_{02}\xi^2 + O(\varepsilon^2)$. We have

$$\lambda^2 + \varepsilon^2\Lambda + O(\varepsilon^3) = 0 \tag{3.16}$$

$$\Lambda = -\frac{1}{4}\frac{\gamma^2 J_0^2 x(3\kappa - 6x + 12x\kappa - 4x^2 + 8x^2\kappa)}{(x+1)(2x+\kappa)} \tag{3.17}$$

where x is the real root of Eq. (3.6). If $\Lambda < 0$, then for sufficiently small ε the characteristic equation (3.16) has a positive root. In this case, by Lyapunov's theorem on stability to a first approximation, orbital instability of the long-period motions occurs. If $\Lambda > 0$, then for sufficiently small ε the roots of the characteristic equation are pure imaginary and we can conclude that the long-period motions are stable in the linear approximation.

Bearing Eqs (3.9) in mind, it can be shown that the numerator of expression (3.17) can only vanish in two cases: either at the boundaries β_1 and β_2, provided that x is a multiple root of Eq. (3.6), or when $x = 0$, where the latter is only possible when $\kappa = 0$. The case $J_0 = 0$, when long-period motions degenerate into an equilibrium position, is not taken into consideration. The denominator of expression (3.17) only vanishes in the special case when $\gamma = 0$, which is not considered in this paper.

Hence, taking into account the continuous dependence of the roots x_i, $(i = 0, 1, 2)$ of Eq. (3.6) on the parameters γ and κ, we can determine the sign of Λ in regions 1–6 (Fig. 1) for each family of long-period motions.

In fact, in the region $0 < \gamma < 1$ the root x_0 depends continuously on the parameters and does not vanish, and hence in this region the values of Λ have the same sign, which can be determined by calculating the value of Λ at an arbitrary point of this region. In the region $\gamma > 1$ the root x_0 also depends continuously on the parameters, but it vanishes when $\kappa = 0$. However, as can easily be seen, the quantity $\Lambda > 0$ does not change its sign when the root x_0 passes through zero, and hence in the region $\gamma > 1$ also the quantity Λ takes values of one sign. Simple calculations show that for $x = x_0$ in the region $0 < \gamma < 1$ the value of Λ is positive, while in the region $\gamma > 1$ (with the exception of $\kappa = 0$) it is negative, and hence the long-period motions corresponding to the root $x = x_0$ are orbitally stable to a linear approximation when $0 < \gamma < 1$ and unstable when $\gamma > 1$, $\kappa \neq 0$. The case $\gamma > 1$, $\kappa = 0$ requires a separate analysis, taking into account terms higher than the fourth power in Hamiltonian (1.2) and is not considered here. Note also that, taking relations (3.8) into account, we can conclude that the long-period motions on the boundary of regions 1 and 2 are orbitally stable: the long-period motions corresponding to the root $x^{(1)}$ are orbitally stable in the linear approximation, while the long-period motions corresponding to the root $x^{(2)}$ are unstable.

Similar discussions show that the long-period motions corresponding to the root $x = x_1$ are orbitally unstable in all regions in which they exist, i.e. in regions 1, 2, 5 and 6, including the boundary separating regions 5 and 6. When $\kappa = 0$ and on the boundaries β_1 and β_2 an investigation of the existence and stability of long-period motions corresponding to the root $x = x_1$, requires that terms higher than the fourth power in Hamiltonian (1.2) should be taken into account and this is not done here. The long-period motions corresponding to the root $x = x_2$ are orbitally stable in all regions in which they exist (in regions 1, 2, 5, 6) and also on the boundary of regions 5 and 6.

4. A non-linear analysis of the orbital stability of long-period motions

If $\Lambda > 0$, then, for a rigorous solution of the problem of the stability of long-period motions it is not sufficient to analyse the linear approximation and it is necessary to investigate the stability taking non-linear terms into account.

We first of all note that in a system with Hamiltonian (3.13), the frequencies of linear oscillations have different orders of smallness. This means that resonances up to the fourth order inclusive are impossible and, by a canonical replacement of the variables $\psi, \rho, \xi, \eta \to \psi_1, \rho_1, \psi_2, \rho_2$, the Hamiltonian (3.13) can be reduced to the normal form

$$\Gamma_* = \Omega_1\rho_1 + \Omega_2\rho_2 + C_{20}\rho_1^2 + C_{11}\rho_1\rho_2 + C_{02}\rho_2^2 + O_3;$$
$$\Omega_2 = \varepsilon\sigma\sqrt{\Lambda} + O(\varepsilon^2), \quad \sigma = -\mathrm{sign}(\upsilon_0) \tag{4.1}$$

The quantities Ω_1 and υ_0 are defined by expressions (3.15) and (3.11) respectively; the expressions for the constant coefficients C_{20}, C_{11}, C_{02} have the form

$$C_{20} = \frac{\varepsilon}{8}\{2b_2d - b_1^2\Lambda^{-1/2}\} + O(\varepsilon^2)$$
$$C_{11} = \frac{\varepsilon}{4}\{b_3d^2 + b_4 - b_1\Lambda^{-1/2}(3h_{03}d^{-1} + h_{21}d)\} + O(\varepsilon^2)$$
$$C_{02} = \frac{\varepsilon}{16}\{6h_{40}d^3 + 2h_{22}d + 6h_{04}d^{-1} - 3\Lambda^{-1/2}(2h_{21}h_{03} + 5h_{03}^2d^{-2} + h_{21}^2d^2)\} + O(\varepsilon^2) \tag{4.2}$$
$$d = (h_{02}/h_{20})^{1/2}$$

The stability of the system with Hamiltonian (4.1) can be analysed using the Arnol'd-Moser theorem, for which it is necessary to verify the condition of isoenergetic non-degeneracy

$$\Delta_2 = C_{20}\Omega_2^2 - C_{11}\Omega_1\Omega_2 + C_{02}\Omega_1^2 \neq 0$$

Using expressions (4.2) and (3.14), after some simplification it can be shown that for sufficiently small ε the condition of isoenergetic non-degeneracy is equivalent to the inequality

$$32(2\kappa-1)^2x^6 + 6(2x+19)(2\kappa-1)^2x^5 + 6(7\kappa+26)(2\kappa-1)^2x^4 +$$
$$+ 6(2\kappa-1)(19\kappa^2+22\kappa-13)x^3 + 6\kappa(23\kappa^2-6\kappa-3)x^2 + 3\kappa^2(11\kappa-3)x + 3\kappa^3 \neq 0 \tag{4.3}$$

We will denote by x the root of Eq. (3.6), corresponding to an orbitally stable family of long-period motions in the linear approximation.

Condition (4.3) was verified numerically. Calculations showed that when $0 < \gamma < 1$ the family of long-period motions, orbitally stable in the linear approximation, corresponding to the root $x = x_0$, will also be orbitally stable in the complete non-linear system. Only the set of points corresponding to the curve α_1 (see Fig. 1), where condition (4.3) is not satisfied, may be an exception. The family of long-period motions, orbitally stable in the linear approximation, corresponding to the root $x = x_2$, will also be orbitally stable in the complete non-linear system, with the exception, perhaps, of the curve α_2 (Fig. 1), where condition (4.3) is not satisfied. To solve the problem of the orbital stability of the long-period motions corresponding to the roots x_0 and x_2 on the curves of α_1 and α_2 respectively, an analysis is necessary taking into account terms no less than the sixth power of the Hamiltonian function (1.2). This problem is not considered in this paper.

5. The orbital stability of periodic motions of a satellite close to its steady rotation

We will consider the motion of a satellite about the centre of mass in a central gravitational field. The satellite is a dynamically symmetrical rigid body, the centre of mass O of which moves in a circular orbit. Suppose Oxyz is a coupled system of coordinates, the axes of which are directed along the principal central axes of inertia of the satellite (the Oz axis coincides with the axis of dynamic symmetry). The axes of the orbital system of coordinates Oxyz are directed along the radius-vector of the centre of mass (OZ), along the transversal (OX) and the binomial to the orbit OY. The orientation of the coupled system of coordinates with respect to the orbital system is specified by the Euler angles ψ, θ, φ.

The equations of motion of the satellite about the centre of mass can be written in canonical form with Hamiltonian[14]

$$H = \frac{p_\psi^2}{2\sin^2\theta} + \frac{p_\theta^2}{2} - p_\psi \operatorname{ctg}\theta\cos\psi - \alpha\beta p_\psi\frac{\cos\theta}{\sin^2\theta} - p_\theta\sin\psi + \alpha\beta\frac{\cos\psi}{\sin\theta} + \frac{\alpha^2\beta^2}{2\sin^2\theta} + \frac{3}{2}(\alpha-1)\cos^2\theta, \quad \alpha\in[0,2] \tag{5.1}$$

The dimensionless momenta, corresponding to the coordinates ψ, θ, φ, are denoted by p_ψ, p_θ, p_φ. The angle φ is the cyclic coordinate, and hence in Eq. (5.1) we will assume that $p_\varphi = \alpha\beta = const$, where $\alpha = C/A$ and $\beta = \Omega_0/\omega_0$, A, B and C ($A = B$) are the principal central moments of inertia, ω_0 is the average motion of the centre of mass in the orbit, Ω_0 is the projection of the angular velocity of the satellite onto its axis of dynamic symmetry, and the parameter β can be any real number.

The canonical system with Hamiltonian (5.1) has the particular solution[15]

$$\psi = \pi, \quad \theta = \pi/2, \quad p_\psi = 0, \quad p_\theta = 0 \tag{5.2}$$

which describes a cylindrical precession, representing steady rotation of the satellite about its axis of dynamic symmetry, situated perpendicular to the orbital plane.

The problem of the stability of cylindrical precession was investigated in detail in Refs 4–17. In Fig. 3 we show a diagram of the stability of the cylindrical precession[14]; the area of instability is shown hatched. In region I the quadratic part of the Hamiltonian of the equations of perturbed motion is positive-definite and the cylindrical precession is stable. In region II the quadratic part of the Hamiltonian of the equations of perturbed motion is alternating. A non-linear analysis of the stability has shown,[14] that on the curve Γ, corresponding to fourth-order resonance, there are two parts of instability of the cylindrical precession. These parts correspond to values of the parameter β from the intervals (0.384642, 0.449337) and (−1.742396, −1.566742). At all other points of the region II the cylindrical precession is stable. Note that on the curve Γ the parameter β takes values in the range $(-\infty, 3/2)$, where points on the curve are uniquely defined by the value of β.

Using the procedure described in Sections 2–4, we analysed the stability of the short-period and long-period motions of the satellite, close to its cylindrical precession, for values of the parameters corresponding to the curve Γ. To do this we expanded Hamiltonian (5.1) in series in the neighbourhood of solution (5.2), we obtained its normal form (1.2),

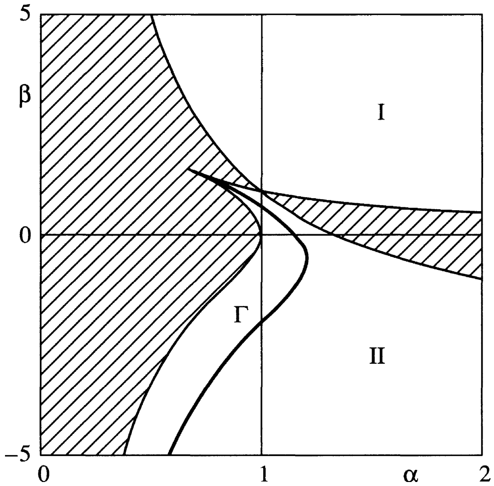

Fig. 3.

and we then carried out calculations using the formulae in Sections 2–4. Omitting the lengthy calculations, we will merely present the final results.

Almost everywhere on the curve Γ the families of short-period motions of the satellite are orbitally stable. Values of the parameter β close to −1.340550 are an exception. In this case, of the stable short-period motions there is one unstable periodic motion which simultaneously belongs both to the family of short-period motions and to the family of orbitally unstable long-period motions. When β = −1.340550, to solve the problem of the stability of the short-period motions it is necessary to analyse terms no lower than the sixth power in the Hamiltonian of the perturbed motion. When β = −2 and β = 2/3 (a spherically symmetrical satellite), the coefficient b of the normal form (1.2) vanishes and as a result Sections 2–4 cannot be used.

An analysis of the stability of the long-period motions showed that when

$$\beta \in (-\infty, -2) \cup (-2, -1.926664) \cup (-1.314341, 0.384642) \cup (2/3, 3/2)$$

on the curve of Γ there is exactly one family of orbitally stable long-period motions. When

$$\beta \in (-1.926664, -1.742396) \cup (-1.566742, -1.340550)$$

$$\cup (-1.340550, -1.314341) \cup (0.449337, 2/3)$$

there are two families of orbitally stable long-period motions and one family of orbitally unstable long-period motions. If

$$\beta \in (-1.742396, -1.566742) \cup (0.384642, 0.449337)$$

then two families of orbitally unstable long-period motions and one family of orbitally stable long-period motions exist. At points of the curve Γ, corresponding to values of the parameter

$$\beta \in \{-1.742396, -1.566742, 0.384642, 0.449337\}$$

there is one family of orbitally stable long-period motions and one family of orbitally unstable long-period motions. At the point corresponding to the value β = −1.340550, there are three families of long-period motions. In this case, on the basis of the results in Sections 3 and 4, one can establish the orbital stability of the long-period motions of two families, while the problem of the orbital stability of the long-period motions of the third family remains open. To solve this it is insufficient to analyse terms of the fourth power of the expansion of the Hamiltonian function in series in the neighbourhood of solution (5.2). For this reason we did not investigate the orbital stability of the long-period motions for β = −1.314341 and β = −1.926664.

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